Generalized Bézier and B-spline patches with exact refinement

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Abstract

In two recent publications we have introduced multi-sided generalizations of Bézier and B-spline surface patches over curved, multiply connected domains. The schemes combine bi-parametric ribbon surfaces; the control points are weighted by Bernstein and/or B-spline basis functions multiplied by rational correction terms that ensure interpolation of the ribbon data. In this paper we propose a new method for setting these rational terms by applying an alternative surface evaluation scheme that algebraically retains the patch interior when boundary ribbons are refined via degree elevation or knot insertion.

1. Introduction

The representation of general, multi-sided surfaces is a fundamental problem in computer-aided geometric design. Such surfaces are needed for designing complex free-form objects, creating smooth transitions that connect a set of given surfaces and reconstructing irregular shapes from point clouds. We have recently proposed a family of genuinely multi-sided parametric surface patches that combine arbitrary interpolant surfaces (*ribbons*) given in Bézier or B-spline form – see the Generalized Bézier[2] (GB) and Generalized B-spline[1] (GBS) patches.

These patches exactly interpolate boundaries and cross derivatives, and can smoothly connect to adjacent tensor-product surfaces. They are defined over a multiply connected, parametric domain with curved boundaries, and thus can interpolate hole loops in the interior of the surfaces. In the equation of these surfaces, the control points of the constituting ribbons are multiplied by rational correction terms, the choice of which is an essential part of the surface definition, and is the subject of this paper.

In order to provide additional degrees of design freedom to the patches, the individual ribbons might need to be refined via degree elevation (GB) or knot insertion (GBS). In our former constructions the boundary constraints were reproduced, but the interior of the patch – due to the rational terms – slightly changed. Our new construction overcomes this deficiency: it preserves the shape interior and provides a more even curvature distribution.

It should be noted that another exact refinement approach was also published recently,[3] where the rational blending functions were treated as part of the control points during the refinement process, retaining the original patch. In comparison, our proposed method is a new, arguably better surface.

In Section 2 we will recall the original patch formulae. In Section 3, after introducing the notion of base patches and reformulating curve and surface equations using displacement vectors, we explain the new construction for degree-elevated G^1 Bézier patches. This is generalized in a fairly straightforward way for patches with G^2 ribbons and for patches with cubic B-spline boundaries (Section 4). Finally in Section 5 a few examples will illustrate the new scheme.

2. Generalized Bézier patches revisited

First we recall the formulation of the Generalized Bézier patch, defined by means of n Bézier ribbon surfaces that determine positions and cross-derivatives along its surface boundaries (see Figure 1). The i-th ribbon $\mathbf{R}_i$ is defined by $(d_i+1)\times(e_i+1)$ control points, given as

$$\mathbf{R}_i(s_i,h_i) = \sum_{j=0}^{d_i}\sum_{k=0}^{e_i}\mathbf{C}^i_{j,k}B^{d_i}_j(s_i)B^{e_i}_k(h_i). \qquad (1)$$

Local ribbon parameters along and across the boundary are denoted by $s_i,h_i\in[0,1]$. The j-th control point in the k-th row $\mathbf{C}^i_{j,k}$ is multiplied by related Bernstein polynomials of the local parameters. Degrees in the longitudinal and cross

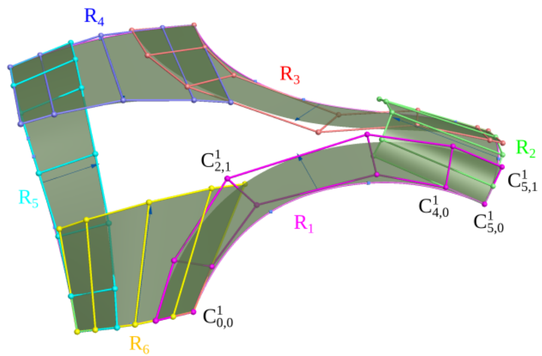

Figure 1: *Ribbons of a GB patch showing some of the control point indices.*

directions (d_i and e_i) can be chosen arbitrarily for each ribbon.

The GB patch is defined over a *curved domain* in the (u,v) plane, and the points of the domain are associated with local bivariate parameters of each ribbon. The side parameter $s_i = s_i(u,v)$ varies linearly on side i between 0 and 1; the distance parameter $h_i = h_i(u,v)$ vanishes on side i and increases monotonically within the domain, eventually reaching 1 on the *distant* sides ($j \neq i-1, i, i+1$). Details of curved domain generation and parameterization can be found in the related papers.

When we formulate the multi-sided patch equation, the control points of ribbon $\mathbf{R}_i$ are retained, but the blending functions need to be modified. We are going to sum up *weighted ribbons* $\mathbf{R}^*_i$ (see Eqs. (6) and (7) later), where in the cross direction we apply Bernstein functions of degree $2e_i+1$ (i) to ensure positional and cross-derivative interpolation with G^{e_i} continuity on side i and (ii) force the weighted ribbon $\mathbf{R}^*_i$ to disappear on the distant sides of the domain ($l \neq i-1, i, i+1$) in both positional and differential sense.

$$\mathbf{R}^*_i(s_i, \overline{h}_i) = \sum_{j=0}^{d_i} \sum_{k=0}^{e_i} \mu^i_j(\overline{h}_i) \cdot \mathbf{C}^i_{j,k} B^{d_i}_j(s_i) B^{2e_i+1}_k(h_i). \tag{2}$$

Each control point is multiplied by a *rational correction term* $\mu^i_j(\overline{h}_i)$ to ensure the interpolation property of the patch. Here $\overline{h}_i$ is an abbreviation for $\{h_{i-1}, h_i, h_{i+1}\}$, i.e., the left and right distance parameters h_{i-1} and h_{i+1} need to be used, as well. The correction terms $\mu^i_j(\overline{h}_i)$ must guarantee that ribbon $\mathbf{R}_i$ is reproduced on the i-th side ($h_i = 0$), and $\mathbf{R}^*_i$ vanishes on the neighboring sides $i-1$ and $i+1$. To satisfy the above properties, $\mu^i_j(\overline{h}_i)$ can be set for the "left and right" corners of the ribbon as

$$\alpha^i = \frac{h_{i-1}^{e_i+1}}{h_{i-1}^{e_i+1} + h_i^{e_i+1}}, \quad j = 0, \ldots, e_i, \tag{3}$$

$$\beta^i = \frac{h_{i+1}^{e_i+1}}{h_{i+1}^{e_i+1} + h_i^{e_i+1}}, \quad j = d_i - e_i, \ldots, d_i; \tag{4}$$

these hold for all k. As we will point out in the next section, the choice of the remaining *middle correction terms* is a delicate issue.

Finally, the multi-sided patch $\mathbf{S}(u,v)$ is defined as the sum of n weighted ribbons. In order to ensure the convex combination property there are two possible solutions. (i) Compute the *normalization sum*, i.e., the sum of all corrected blending functions*:*

$$B_\Sigma(u,v) = \sum_{i=1}^{n} \sum_{j=0}^{d_i} \sum_{k=0}^{e_i} \mu^i_j(\overline{h}_i) B^{d_i}_j(s_i) B^{2e_i+1}_k(h_i), \tag{5}$$

and divide the patch equation by this sum

$$\mathbf{S}(u,v) = \frac{1}{B_\Sigma(u,v)} \cdot \sum_{i=1}^{n} \mathbf{R}^*_i(s_i, h_i). \tag{6}$$

(ii) Compute the *weight deficiency* as $1 - B_\Sigma(u,v)$ and assign this blending function to either a single "central" control point $\mathbf{C}_0$ or an arbitrary "central surface" $\mathbf{C}_0(u,v)$ defined over the same domain. Then

$$\mathbf{S}(u,v) = \sum_{i=1}^{n} \mathbf{R}^*_i(s_i, h_i) + \mathbf{C}_0(1 - B_\Sigma(u,v)). \tag{7}$$

The latter solution might be beneficial when we want to modify the shape of the patch interior, see e.g. Figure 4.

3. Degree elevation and the displacement-based evaluation

We are going to present a new variant of Generalized Bézier patches, but – for simplicity's sake – we show our concept by means of a degree-d ($d \geq 3$) Bézier curve $\mathbf{r}(s)$, being elevated to degree D. The new control points $\mathbf{C}^D_i$ are linear combinations of the $\mathbf{C}^d_i$-s, and the curve remains unchanged, i.e.,

$$\mathbf{r}(s) = \sum_{j=0}^{d} \mathbf{C}^d_j B^d_j(s) = \sum_{j=0}^{D} \mathbf{C}^D_j B^D_j(s). \tag{8}$$

Now let us take a *weighted* Bézier curve, where the control points are multiplied by some additional scalar weights μ^d_j. Assume that we have fixed weights for the first two and the last two control points, denoted by α and β, respectively, and we have freedom to set the remaining weights in the middle $\left\{\mu^D_j\right\}_{j=2\ldots D-2}$. We would like to degree-elevate the original control points by the standard rules *and* obtain an

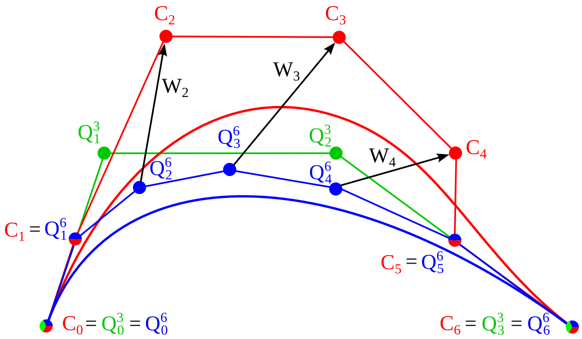

Figure 2: *Bézier curve: degree elevated base curve + displacements.*

identical equation. In general this does not hold:

$$\mathbf{r}^*(s) = \sum_{j=0}^{d} \mu_j^d \mathbf{C}_j^d B_j^d(s) \neq \sum_{j=0}^{D} \mu_j^D \mathbf{C}_j^D B_j^D(s). \tag{9}$$

The formerly proposed methods of (i) setting the mid-weights to constant 1 (*mid-1* scheme) or (ii) applying a linear combination[2] of α and β are not satisfactory. For example, let us elevate from cubic ($d = 3$), $\mu^d = [\alpha,\alpha,\beta,\beta]$ to quartic ($D = 4$), and try either $\mu^D = [\alpha,\alpha,1,\beta,\beta]$ or $\mu^D = [\alpha,\alpha,\frac{\alpha+\beta}{2},\beta,\beta]$. As we degree-elevate, the left side of the above equation is multiplied by $[(1-s)+s]$ leading to the central term $\frac{1}{2}(\alpha C_1^3 + \beta C_2^3)$, which is *not* equal to $\mu_2^4 C_2^4 = \mu_2^4 \cdot \frac{1}{2}(C_1^3 + C_2^3)$ for any choice of μ_2^4, unless $\alpha = \beta$.

For solving this problem, first we describe an alternative approach that allows correct degree elevation of weighted curves. The key idea is to reformulate the equation of Bézier curves using *displacements* over degree-reduced *cubic base curves,* see Figure 2. Take an arbitrary degree-D Bézier curve $\mathbf{r}(s)$ with control points $\left\{\mathbf{C}_j^D\right\}_{j=0\ldots D}$ (red curve, $D = 6$), then construct a corresponding cubic base curve with control points $\left\{\mathbf{Q}_j^3\right\}_{j=0\ldots 3}$, determined by the endpoint positions and first derivatives (green curve). Next degree-elevate the base curve, obtaining control points $\left\{\mathbf{Q}_j^D\right\}_{j=0\ldots D}$ (blue curve), and compute displacement vectors as $\left\{\mathbf{W}_j^D = \mathbf{C}_j^D - \mathbf{Q}_j^D\right\}_{j=0\ldots D}$. By the above definition of the base curve $\mathbf{W}_0^D = \mathbf{W}_1^D = \mathbf{W}_{D-1}^D = \mathbf{W}_D^D = \mathbf{0}$, and thus

$$\mathbf{r}(s) = \sum_{j=0}^{D} \left(\mathbf{Q}_j^D + \mathbf{W}_j^D\right) B_j^D(s) \tag{10}$$

$$= \sum_{j=0}^{D} \mathbf{Q}_j^D B_j^D(s) + \sum_{j=0}^{D} \mathbf{W}_j^D B_j^D(s) \tag{11}$$

$$= \sum_{j=0}^{3} \mathbf{Q}_j^3 B_j^3(s) + \sum_{j=2}^{D-2} \mathbf{W}_j^D B_j^D(s). \tag{12}$$

We can perform the same reformulation on a degree-elevated weighted curve $\mathbf{r}^*(s)$:

$$\mathbf{r}^*(s) = \sum_{j=0}^{3} \left\{ \begin{array}{l} \alpha, j = 0,1 \\ \beta, j = 2,3 \end{array} \right\} \mathbf{Q}_j^3 B_j^3(s) + \sum_{j=2}^{D-2} \mathbf{W}_j^D B_j^D(s). \tag{13}$$

We have separated the weighted terms of the base curve from those terms that arise due to degree elevation, yielding an expression that defines a curve identical to the original one when there are no displacements. The $\mathbf{W}_j^D$ displacements are represented by vectors and thus do not change the normalization sum. Formally, the control points $\mathbf{C}_j^D$ and $\mathbf{Q}_j^D$ appearing in the equation for $\mathbf{W}_j^D$ are multiplied by the same blending function $B_j^D(s)$ with opposite signs, which cancel each other. The weighted terms associated *only* with the base curve are independent of degree D.

We apply the same displacement-based evaluation to the weighted G^1 ribbons of the multi-sided GB patch:

$$\mathbf{R}_i^*(s_i,\bar{h}_i) = \sum_{k=0}^{1} \left(\sum_{j=0}^{3} \left\{ \begin{array}{l} \alpha^i(\bar{h}_i), j = 0,1 \\ \beta^i(\bar{h}_i), j = 2,3 \end{array} \right\} \mathbf{Q}_{j,k}^3 B_j^3(s_i) + \sum_{j=2}^{d_i-2} \mathbf{W}_{j,k}^d B_j^d(s_i) \right) \sum_{j=2}^{d_i-2} \mathbf{W}_{j,k}^d B_j^d(s_i) \tag{14}$$

In our context, the equation naturally reproduces all degree-elevated ribbons on the boundary, where $h_i = 0$ and $\alpha(\bar{h}_i) = 1$ and $\beta(\bar{h}_i) = 1$. Since the weighted ribbons and the sum of the weighted blending functions are retained, the whole patch remains identical. In fact, for a fixed domain and local parameterization, the cubic base patch and the corresponding normalization term (and weight deficiency) are uniquely determined, being independent of the degrees of the ribbons:

$$B_\Sigma(u,v) = \sum_{i=1}^{n} \sum_{j=0}^{3} \sum_{k=0}^{1} \left\{ \begin{array}{l} \alpha^i(\bar{h}_i), j = 0,1 \\ \beta^i(\bar{h}_i), j = 2,3 \end{array} \right\} B_j^3(s_i) B_k^3(h_i). \tag{15}$$

The above idea can easily be generalized for GB patches with G^2 continuity: then the boundaries need to be degree 5 or higher, with three rows of control points ($e_i = 2$) and quintic Bernstein functions in the cross direction. The base patch will be composed of ribbons with quintic boundaries, and at the corners the α^i and β^i weights will be applied to 3×3 instead of 2×2 control points.

4. Generalized B-spline patches revisited

GBS patches are similar to GB patches, having B-spline boundaries in the longitudinal direction.[1] The ribbons have $(d_i + 1) \times (e_i + 1)$ control points, multiplied by a 'hybrid' product of basis functions $N_j^{\xi_i,p_i}(s_i) B_k^{e_i}(h_i)$, where $N_j^{\xi_i,p_i}$ are B-spline functions defined over a knot vector ξ_i with degree p_i. The degrees, the knot vectors and the number of cross-derivatives can be different for each B-spline ribbon. In all

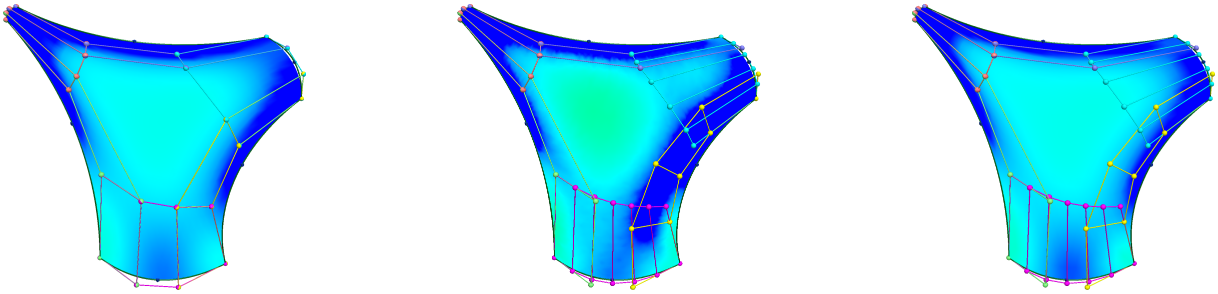

Figure 3: *Degree elevation (showing curvatures). Left: cubic boundaries, middle:* mid-1 *changes the shape, right: displacement retains the shape.*

equations for GB patches in Sections 2 and 3 the Bernstein basis functions can be replaced by B-splines in the longitudinal direction.

Our problem is the same as before: retain the weighted ribbons – and thus the multi-sided patch – when new knots are inserted. The solution is also similar: define *weighted* B-spline ribbons by creating a base patch and applying displacements. As an example, we create a cubic B-spline ribbon with G^1 constraints at the corners. The control points are denoted by $\{\mathbf{C}_{j,k}\}_{j=0\dots d;k=0,1}$. We deal with clamped B-splines, so the knot vector is $\xi_i = [0,0,0,0,\nu_1,\dots,\nu_{K_i},1,1,1,1]$, where K_i denotes the number of internal knots and $K_i = d_i - 3$. We remove all the internal knots, preserving the first derivatives at the ends; this yields a cubic base ribbon with control points $\{\mathbf{Q}_{j,k}\}_{j=0\dots 3;k=0,1}$ and knot vector $\hat{\xi}_i$. After reinserting the internal knots, we create an identical ribbon with control points $\left\{\mathbf{Q}^*_{j,k}\right\}$ and displacement vectors $\left\{\mathbf{W}_{j,k} = \mathbf{C}_{j,k} - \mathbf{Q}^*_{j,k}\right\}$, where $j = 0\dots d$ and $k = 0,1$. In this way we separate the weighted corner quantities from the displacement term in the middle and the weighted ribbons will remain intact during knot insertion, i.e.,

$$\mathbf{R}^*_i(s_i,\bar{h}_i) = \sum_{k=0}^{1}\left(\sum_{j=0}^{3}\left\{\begin{array}{l}\alpha^i(\bar{h}_i),\, j=0,1\\ \beta^i(\bar{h}_i),\, j=2,3\end{array}\right\}\mathbf{Q}^i_{j,k}N_j^{\xi_i,3}(s_i)+ \sum_{j=1}^{K_i}\mathbf{W}^i_{j,k}N_j^{\hat{\xi}_i,3}(s_i)\right)B^3_k(h_i). \qquad (16)$$

The construction for G^2 GBS patches with cubic B-spline boundaries is somewhat more complicated, as we need at least 6 degrees of freedom in the longitudinal direction to separate the first and second derivative constraints at the two corners. This will lead to a base patch where each boundary has three segments. Finally the actual knots can be reinserted and the displacement vectors computed accordingly.

As it was discussed in the paper,[1] GBS patches can also handle hole loops given by periodic B-spline ribbons, using a special parameterization. The surface equations in this case are also computed as the sum of weighted ribbons. But as there are no correction terms for periodic ribbons, knot insertions trivially preserve the full multi-sided patch.

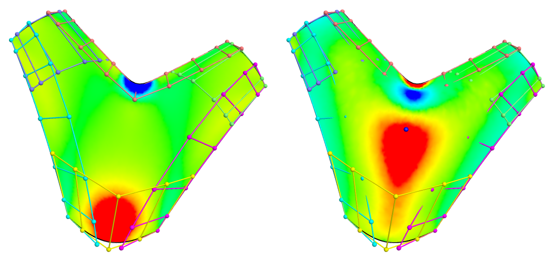

Figure 4: *Interior shape editing (mean curvature map).*

5. Discussion

To ensure an invariable patch interior after degree elevation or knot insertion was a theoretical challenge for this sort of multi-sided patches. In Figure 3 a 6-sided GB patch can be seen (left); we have degree-elevated various ribbons and the shape has slightly changed (middle), but the image on the right shows that the displacement-based evaluation keeps the shape unchanged. We remark that in practical situations the user generally refines the control structure in order to relocate the control points or compute some better approximation, so the patch interior is likely to change in all these events.

We have found that formulating the weighted ribbons by displacements is beneficial for the overall shape due to a more balanced curvature distribution. This is mainly due to the normalization term/weight deficiency functions defined over the curved domain. Cubic base patches produce natural weight deficiencies with minimal oscillation, while the previous weighting methods often produced uneven distributions with dominantly negative values, where the sum of the weighted blending functions exceeded 1. The differences in the final surface are minor when using normalization (6), but may become important when the weight deficiency is utilized for adjusting the interior – see a simple example in Figure 4.

Figure 5 shows an example, where we compare a GBS patch by the old method (*mid-1*) and the new displacement-based formulation. It is not easy to distinguish between the two surfaces, but the related weight deficiency functions – shown by contoured values elevated over the curved domain – significantly differ. Another example of a multiply connected GBS patch is shown in Figure 6, where similar observations can be made. We remark that the use of displacement vectors yields an affine combination of the original control points of the ribbons.

We also note that the displacement scheme reproduces the classical four-sided Coons patch using the given ribbons over a rectangular domain.

Conclusion

We have modified the equation of generalized Bézier (GB) and B-spline (GBS) patches to ensure invariance when the ribbons are refined by degree elevation or knot insertion. The new formula is also beneficial concerning the normalization term/weight deficiency functions, which are inherent features of this sort of multi-sided patches.

Acknowledgments

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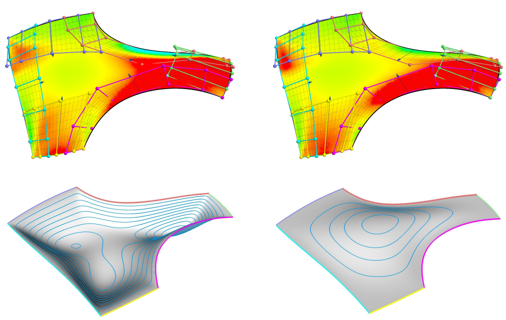

Figure 5: *Comparing patches (mean curvature map) and weight deficiency functions (*mid-1 *vs. displacement scheme).*

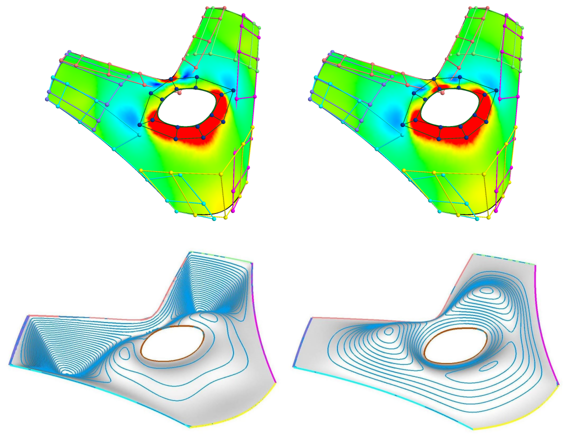

Figure 6: *Comparing multiply connected patches (mean curvature map) and weight deficiency functions (*mid-1 *vs. displacement scheme).*