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Generalized B-spline surfaces over curved domains

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Abstract

We propose a new surface representation, the Generalized B-spline (GBS) patch, that combines ribbon interpolants given in B-spline form. A GBS patch can connect to tensor-product B-spline surfaces with arbitrary G^m continuity. It supports interpolation constraints not only along the outer perimeter loop, but also around holes in the interior of the patches.

Ribbon control points are weighted by products of B-spline and Bernstein basis functions, multiplied by rational terms ensuring geometric continuity. A new local parameterization method is introduced using harmonic functions, that handles periodic hole loops, as well. Several examples illustrate the capabilities of the proposed scheme. This paper is a shortened version of a recent journal publication by Vaitkus et al.[26]

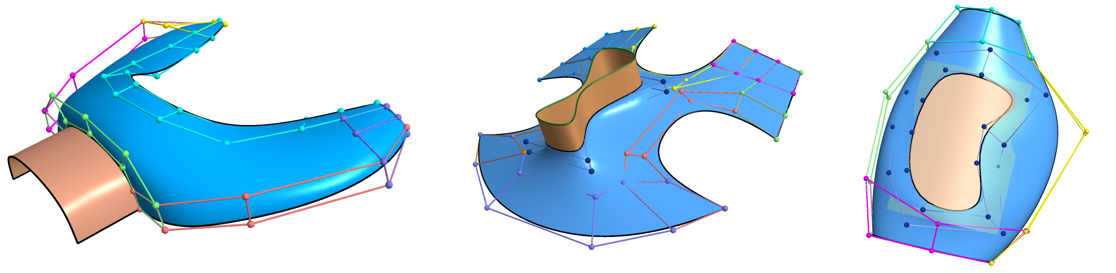
1. Introduction

The representation of general, *multi-sided surfaces* is a fundamental problem in Computer-Aided Geometric Design. Such surfaces are needed for designing and reconstructing complex free-form objects, and for creating smooth transitions that connect a set of *given* surfaces (hole filling, vertex blending, lofting). Multi-sided surfaces can be generated either by *trimming* or adjoining standard four-sided (tensor product Bézier or B-spline) patches, or alternatively as *genuinely multi-sided* parametric patches, defined over some non-rectangular two-dimensional region. While four-sided patches are compatible with industrial CAD standards, multi-sided patches generally allow for higher quality surfaces, as well as *precise boundary control*. However, problems emerge in design situations where the boundary curves are concave with high curvature variation, and explicit boundary and cross-derivative constraints need to be satisfied along hole loops of a multiply connected surface.

To overcome these problems we propose a new multi-sided surface representation, the *Generalized B-spline (GBS)* surface, that connects to ribbon interpolants given in tensor-product B-spline form with arbitrary G^m continuity. GBS patches are defined by control points, weighted by products of B-spline and Bernstein basis functions, pulled back from their standard rectangular domain onto a general subset of the plane using local (re-)parameterizations. The weight functions are further blended using rational terms to ensure smooth connections with the interpolants.

Choosing the parametric domain and the local parameterizations are important aspects of any multi-sided surface scheme. The majority of existing multi-sided parametric surfaces are defined over regular polygonal domains, which affects the geometric quality of the surfaces that can be modeled, similarly to uniform knot vectors in traditional NURBS representations. To facilitate the modeling of complex shapes, GBS patches are defined over *curved*, possibly *multiply connected* planar subsets. The *local parameterization* of such general domains pose a new challenge, which we tackle using a novel parameterization method based on *harmonic functions*, that can easily be generalized also to *periodic parameterization* of interior loops.

We show three simple examples illustrating the benefits of the GBS surface representation. The first model is bounded by a winding, concave ribbon, and another representing a planar profile on an extruded surface, see Figure 1a. It would be difficult to produce such a constrained patch by trimming; in contrast, the GBS patch yields a nice shape. On the second model (Figure 1b), the base surface has several concave boundaries as well as a periodic B-spline ribbon in the interior with vertical cross-derivatives. Traditional CAD would require complex trimming and lofting operations to reach this configuration, while the GBS patch produces a nice, multiply connected shape without auxiliary steps. The third example is a half bottle (Figure 1c) with an interior ribbon matching a developable surface label, blended with the ribbons of the bottle using a single GBS patch.



(a) Highly curved concave boundary (b) Interior loop with cross-derivatives (c) Half-bottle with label surface

Figure 1: Motivational examples of Generalized B-spline surfaces.

The paper is structured as follows. A brief review of relevant prior art is given in Section 2, followed by a detailed description of the GBS patch equations in Section 3. We present our novel methods for the generation and local parameterization of curved domains in Section 4. We demonstrate the modeling capabilities of GBS patches via a variety of case studies in Section 5. Finally, we draw conclusions and discuss opportunities for future work.

2. Previous work

Our primary focus here is the representation of multi-sided surfaces. These can be categorized as (i) smoothly connected macropatches composed of quadrilaterals, (ii) transfinite interpolation and (iii) control point based representations. Peters[16] and Hughes et al.[10] exhaustively survey the variety of macropatch approaches, while a good summary of transfinite schemes can be found in Várady et al.[27] Very few methods have the ability to model multiply connected surface patches – we are only aware of the attempts by Kato[12] and Sabin,[20] both in the transfinite setting.

We restrict the rest our discussion to control point based methods. Concerning those that generalize the Bernstein–Bézier representation, there are a variety of classical approaches, including S-patches,[14] toric patches,[13] Zheng-Ball patches[32] and M-patches,[11] see also Chapter 8 of the textbook[3] by Goldman. Our work is most closely related to the *Generalized Bézier (GB) patches* introduced by Várady et al.[28, 29] Recent advances on GB patches included the removing of twist compatibility assumptions,[6] and generalizations to concave,[23] and then curved domains.[30]

There are also control point based, multi-sided patches bounded by B-spline curves. The work of Pla-Garcia et al.[19] maps B-spline basis functions to a planar spline domain with fixed knot vectors. Similarly, Yuan and Ma[31] maps oriented tensor-product B-spline functions to vertices of a flattened quad mesh. The recent works of Hettinga and Kosinka[8, 7] describe a hole filling approach at the irregular points of subdivision surfaces, extending the convex GB patch[28] using modified blending functions to ensure G^2 continuity with surrounding surface elements. Isogeometric finite element analysis also pulls back B-spline functions onto a general domain through a piecewise-polynomial parameterization.[10]

In a recent preprint Martin and Reif[15] introduced *ABC patches* for the accurate boundary control of trimmed NURBS patches. The domain and the local parameterizations are defined implicitly by B-spline functions, and the result can be converted into a collection of standard NURBS surfaces.

After the publication of our original journal paper, a fresh work of Sabin et al.[21] applied perturbations to B-spline patches along *curved knot lines* in a CAD-compatible way. The domain boundaries must be implicit lines or conics, and the perturbations must be expressed as constant jumps in derivatives.

The GBS patch proposed in this paper lifts most (implicit or explicit) assumptions of previous representations. It does not require uniform or symmetric knot arrangements, and combines arbitrary B-spline ribbons with arbitrary geometric continuity. while being capable of handling complex concave boundaries and hole loops as well.

3. GBS patch equation

The GBS patch is defined by n piecewise-polynomial *ribbons*, determining positions and cross-derivatives along subsets of the surface boundary. We distinguish *open* (clamped) and *closed* (periodic) B-spline ribbons: open ribbons connect sharp corners along the boundary; closed ribbons define hole loops in the interior. The i-th ribbon R_i is defined by $(p_i+1)\times(m_i+1)$ control points, and its equation is given in the following form:

$$\mathbf{R}_i(s_i,h_i)=\sum_{j=0}^{p_i}\sum_{k=0}^{m_i}\mathbf{C}^i_{j,k}\cdot N_j^{\xi_i,d_i}(s_i)B_k^{m_i}(h_i). \qquad (1)$$

Local ribbon parameters along and across the boundary are denoted as $s_i,h_i\in[0,1]$. The j-th control point in the k-th row $\mathbf{C}^i_{j,k}$ is multiplied by B-spline functions $N_j^{\xi_i,d_i}$ with knot vector ξ_i and degree d_i, and Bernstein polynomials $B_k^{m_i}$. Note that the degrees and knot vectors, as well as the number of cross-derivatives could be different for each B-spline ribbon.

The GBS patch is defined over a *curved domain* in the (u,v) plane, and each side corresponds to a local ribbon *(re-)parameterization*, where the side parameter $s_i = s_i(u,v)$ varies linearly on side i between 0 and 1, while the distance parameter $h_i = h_i(u,v)$ vanishes on side i and increases monotonically within the domain, eventually reaching 1 as it gets to the 'distant' sides. A multiply connected domain with its local parameterizations are shown in Figure 2. Details of domain generation and parameterization will be given in Section 4.

When formulating the patch equation we combine modified ribbons $\mathbf{R}_i^*$, using degree elevated Bernstein functions in the cross direction. We also apply *rational correction terms* to ensure that the multi-sided patch interpolates the original ribbons with G^{m_i} continuity:

$$\mathbf{R}_i^*(s_i,h_i) = \sum_{j=0}^{p_i}\sum_{k=0}^{m_i} \mathbf{C}_{j,k}^i \cdot \mu_j^i(h_i) N_j^{\xi,d_i}(s) B_k^{2m_i+1}(h). \quad (2)$$

where correction terms $\mu_j^i(h_i)$ (which also depend on the distance parameters of the adjacent sides h_{i-1} and h_{i+1}) are similar to Gregory's square:[5]

$$\mu_j^i = \begin{cases} h_{i-1}^{m_i+1}/(h_{i-1}^{m_i+1}+h_i^{m_i+1}), & j=0,\dots,m_i \\ h_{i+1}^{m_i+1}/(h_{i+1}^{m_i+1}+h_i^{m_i+1}), & j=p_i-m_i,\dots,p_i \\ 1, & \text{otherwise} \end{cases} \quad (3)$$

The sums of uncorrected and corrected blend functions over a multiply connected domain are shown in Figure 3.

Finally, in order to ensure the convex combination property, we normalize the patch equation:

$$\mathbf{S}(u,v) = \frac{1}{B_\Sigma(u,v)} \cdot \sum_{i=1}^{n} \mathbf{R}_i^*(s_i,h_i), \quad (4)$$

by dividing with the weight sum

$$B_\Sigma(u,v) = \sum_{i=1}^{n}\sum_{j=0}^{p_i}\sum_{k=0}^{m_i} \mu_j^i(h_i) N_j^{\xi,d_i}(s_i) B_k^{2m_i+1}(h_i)). \quad (5)$$

An alternative to normalization is to assign the weight deficiency $1 - B_\Sigma(u,v)$ to a single "central" control point. In certain cases it might be beneficial to introduce additional control points that only affect the patch interior. We refer to Section 5 of our journal paper[26] for further discussion on the various possibilities of interior shape control.

The above formulas can be generalized in a straightforward manner for multi-loop configurations as well, but note that rational correction terms are not necessary for closed/periodic ribbons – see Figure 3c.

Generalized Bézier patches over curved domains[30] are special GBS patches, where every B-spline consists of a single Bézier segment – see also Section 5.6.

4. Domain and parameterization

In the following we describe our methods for curved domain generation and local parameterization. In this paper we only give a high-level description of our implementations – further technical details can be found in Section 4 of our journal paper[26].

4.1. Domain generation

The curved domains of GBS patches are generated by sampling the boundary curves into polylines, then *developing* them into the tangent planes of the interpolants, meaning the curves are isometrically flattened while preserving their shape within the surface (i.e., their *geodesic curvature*). The developed 2D polylines generally do not form a closed loop (due to the non-zero Gaussian curvature of the surface patch), thus their vertices are shifted to a minimal extent such that they do so.

The interior loops of multiply connected surfaces are projected onto the GBS patch defined by the outer perimeter loop.

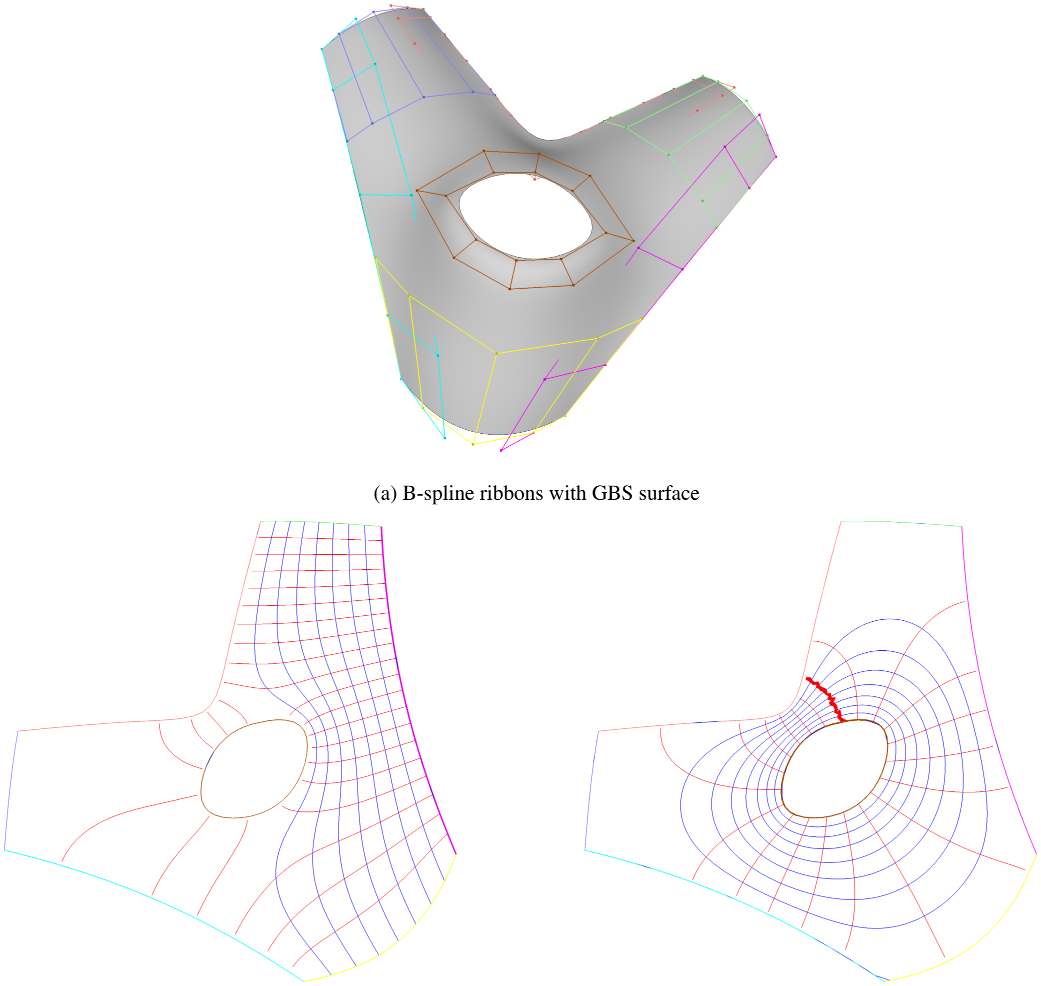
4.2. Local parameterization

Our goal is to determine local $(s_i(u,v), h_i(u,v))$ parameters for each ribbon over the curved domain $\Omega \subset \mathbb{R}^2$ that satisfy the properties described in Section 3. We look for *harmonic functions*, which are constrained minimizers of the Dirichlet energy:

$$\begin{aligned} \underset{f}{\text{minimize}} \quad & \int_D |\nabla f|^2 \, dA \\ \text{subject to} \quad & f(x) = b_f(x), x \in \mathcal{D}_f, \end{aligned} \quad (6)$$

where $f = s_i$ or h_i, and b_f encodes Dirichlet boundary conditions for the constrained subset of the boundary $\mathcal{D}_f \subset \partial\Omega$. Harmonic functions are preferred, as they are guaranteed to be C^∞-continuous in the interior as well as monotonic without local maxima, and are also widely used in various applications,[2, 4] including tensor-product reparameterizations of curved domains.[9] The method can also be generalized to periodic parameterization of closed ribbons by prescribing quantized jumps for the s-coordinate value along a series of (virtual) cuts.

This variational problem can easily be discretized over a triangulation of the domain, resulting in a discrete Laplace equation.[18] The parameterization is then given in the form of a C^0 piecewise-linear function. Note that the surface will eventually be tessellated for rendering and downstream applications anyway, and higher-order approximations to harmonic parameterizations could also be computed using alternative methods[24, 10].



(a) B-spline ribbons with GBS surface

(b) Curved domain with local parameters

(c) Interior loop

Figure 2: Multiply connected curved domain and local parameterizations for a GBS patch.

5. Case studies

In this section we discuss some interesting features of GBS patches, and show related case studies. In our examples, the boundary curves and cross-derivatives are defined by *non-uniform, cubic B-splines*. In the hole filling context ribbons are explicitly derived from the surrounding patches. In curve network based design the adjacent ribbons are specified to match a common direction blend or curvature function, computed over their common knot sequence. We use an approximate G^1/G^2 connection algorithm[22], where ribbons match a common rotation minimizing frame.

5.1. Convex vs. curved domains

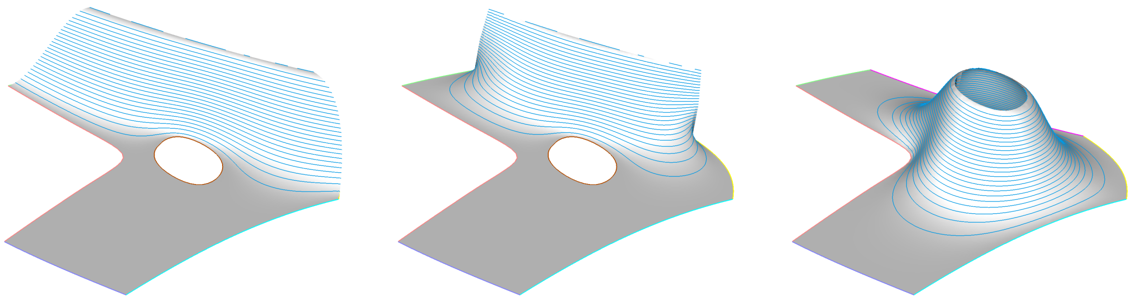
Classical methods for modeling complex multi-sided regions often fail, and the shape may fold under itself. On Figure 4 we compare a convex-domain Charrot–Gregory patch[1] with a curved-domain GBS patch.

5.2. Sheet metal part

A surface model defined by a concept car sketch are shown on Figure 5, together with curvature map. The control structure contains two ribbons with three layers for additional shape control.

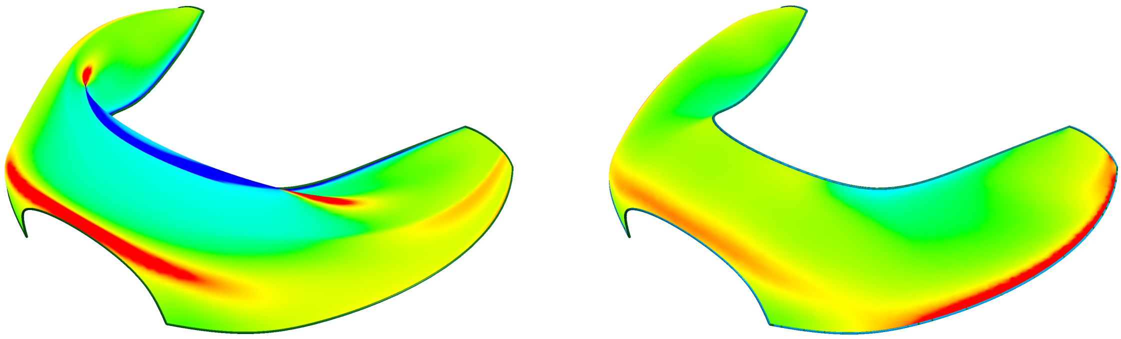
5.3. Smoothly connected patch network

Figure 6 shows a model for the front part of a concept car, together with slicing and a mean curvature map. Four GBS patches are smoothly connected with G^1 continuity.



(a) Before correction (b) After correction (c) Interior loop

Figure 3: Sum of ribbon B-spline basis functions pulled back onto a multiply connected domain.



(a) Charrot–Gregory patch, convex domain (b) GBS patch, curved domain

Figure 4: Comparison of convex and curved domains. Colors indicate mean curvature.

5.4. Multi-sided patches at extraordinary vertices

One important utilization of multi-sided patches is to fill holes at the extraordinary vertices of quadrilateral meshes, for example in Catmull–Clark subdivision. The quality of these patches is particularly important, and various solutions have been published[16, 8]. The key idea is to borrow positional and cross-derivative information from the surrounding regular patches and insert a multi-sided patch with G^2 continuity – for example, a GBS patch with suitable three-layer ribbons. We have picked an example from a standard test suite[17] and examine the curvature map and the distribution of isophotes around the extraordinary point on Figure 7.

5.5. Multiply connected surface

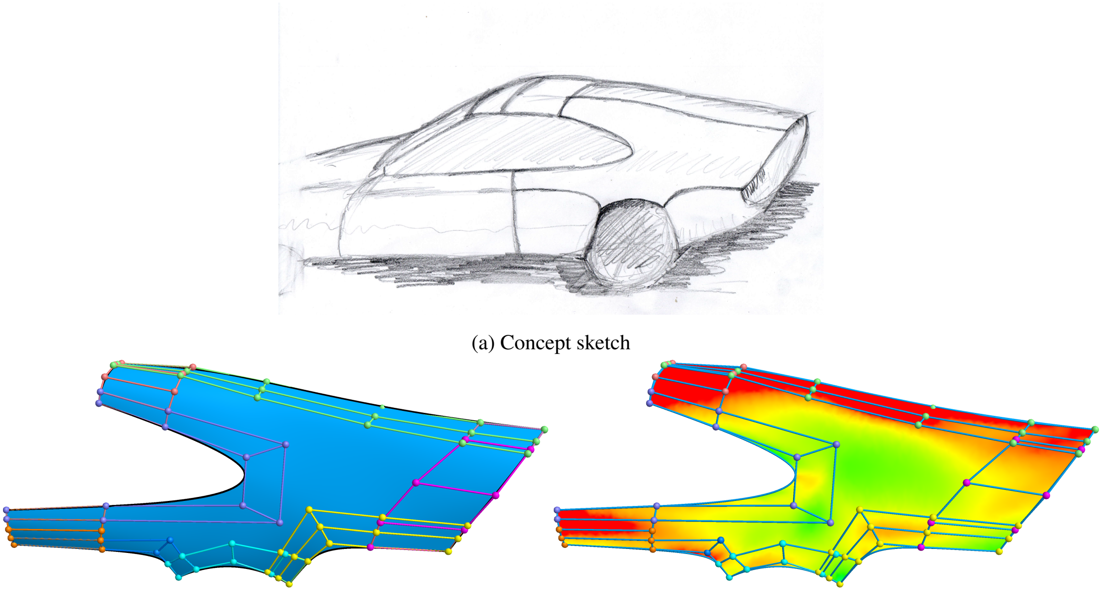
We illustrate the ability of GBS patches to model multiply connected surfaces using the example of a plastic bottle, with Figure 8 showing the original surface and its modification with an interior loop. The cross-derivatives for the side ribbons and the hole loop must be orthogonal to the symmetry plane of the bottle in order to smoothly connect with the other half.

5.6. Polyhedral design

GBS patches are also well-suited to represent a patch complex indirectly defined by a control polyhedron and an associated curve network. We refer to a companion paper in the same conference proceedings[25] where curved domain Bézier patches are used in polyhedral-based design.

6. Conclusions and future work

We have proposed a new multi-sided surface representation that interpolates complex B-spline boundary curves with associated cross-derivatives as well as periodic B-splines in the interior of multiply connected patches. The basic elements of the construction include curved domain generation, harmonic local parameterization for multiply connected domains, and a combination of mapped B-spline and Bernstein basis functions with rational correction terms. We have shown a variety of interesting shapes being represented by GBS patches. We believe that these can hardly be produced by means of trimming or macro-patch methods, or even traditional multi-sided schemes based on convex polygonal domains.



(a) Concept sketch

(b) Control structure

(c) Mean curvature map

Figure 5: A panel of a sports car.

6.1. Future work

One particular issue is the weight deficiency in the interior of the patches, which we currently correct by normalization, the effect of which can be difficult to predict. Constructing a uniform structure of interior control points for GBS patches – similar to that of convex GB patches[28] – appears to be a promising research programme.

Another problem is that harmonic h-parameter lines are unevenly distributed for long, strongly curved boundaries. Resolving this would ensure that the ribbon cross-derivatives affect the interior to a uniform extent. We also plan to explore alternative methods for computing harmonic functions, such as Monte Carlo,[24] and higher-order isogeometric finite elements.[10]

We further remark that designing appropriate B-spline ribbons may become difficult in certain situations, thus interactive CAD operations that support the designers' work would facilitate the wider use of GBS patches.

Finally, we realize that the primary limitation of the GBS representation is its lack of CAD compatibility. We are currently actively researching methods for approximating GBS patches with a network of standard NURBS surfaces.

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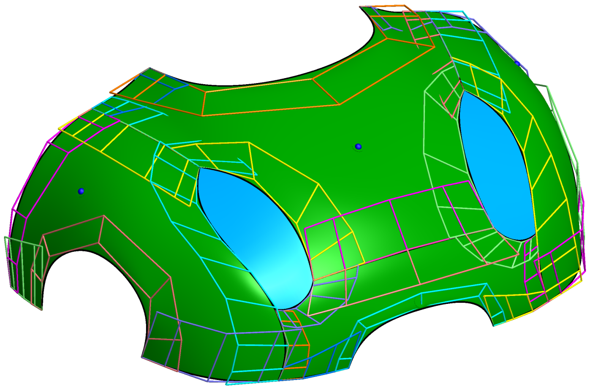
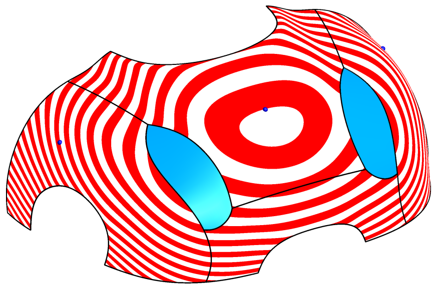
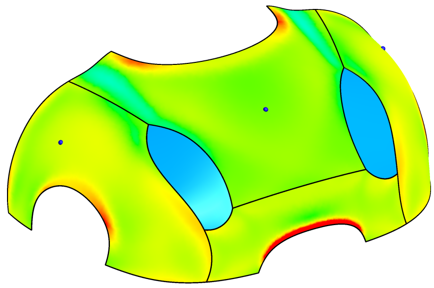


(a) Ribbons and GBS surfaces



(b) Slicing



(c) Mean curvature

Figure 6: Model with G^1-connected GBS patches.

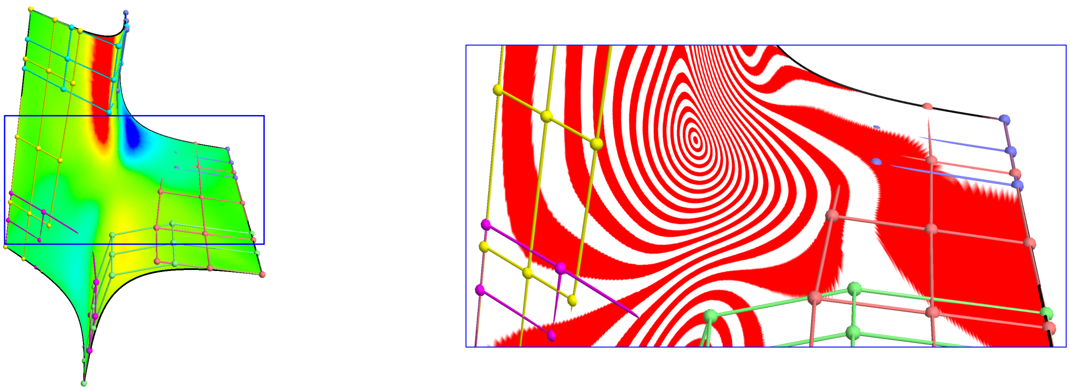


Figure 7: GBS patch as G^2 cap in subdivision surfaces with mean curvature map and isophote lines near the center.

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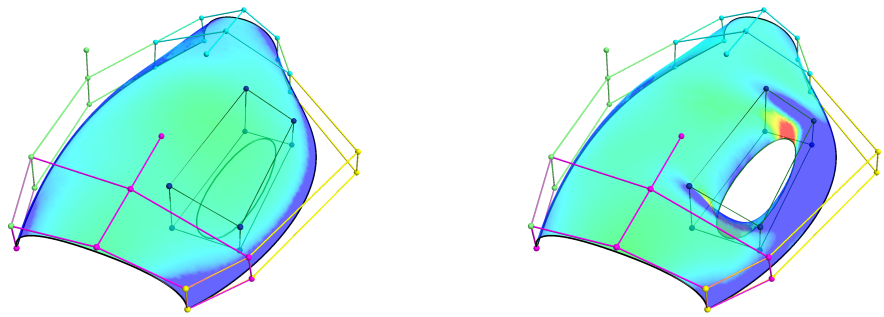

(a) Without the hole ribbon

(b) With the hole ribbon

Figure 8: Multiply connected half-bottle surface. Colors indicate mean curvature.

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