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A multi-sided generalization of the C^0 Coons patch

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Abstract

Most multi-sided transfinite surfaces require cross-derivatives at the boundaries. Here we show a general n-sided patch that interpolates all boundaries based on only positional information. The surface is a weighted sum of n Coons patches, using a parameterization based on Wachspress coordinates.

1. Introduction

Filling an n-sided hole with a multi-sided surface is an important problem in CAGD. Usually the patch should connect to the adjacent surfaces with at least G^1 continuity, but in some applications only positional (C^0) continuity is needed, and normal vectors or cross derivatives at the boundary curves are not available.

For $n = 4$, the C^0 Coons patch[1] solves this problem; in this paper we show how to generalize it to any number of sides.

2. Previous work

Most transfinite surface representation in the literature assume G^1 constraints, and the patch equations make use of the fixed cross-derivatives at the boundary. This can be circumvented by generating a *normal fence* automatically, e.g. with a rotation minimizing frame[3]; however, in a C^0 setting this is an overkill, simpler methods exist.

One well-known solution is the harmonic surface, which creates a "soap film" filling the boundary loop by solving the harmonic equation on a mesh with fixed boundaries. This, however, minimizes the total area of the surface, which often has unintuitive results, see an example in Section 4.

The basic idea of the proposed method, i.e., to define the surface as the weighted sum of n Coons patches, each interpolating three consecutive sides, is the same as in the CR patch[2].

3. The multi-sided C^0 Coons patch

Let $C_i(t) : [0,1] \to \mathbb{R}^3$ denote the i-th boundary curve. Let us also assume $C_i(1) = C_{i+1}(0)$ for all i (with circular indexing). Then the *ribbon* R_i is defined as a C^0 Coons patch interpolating C_{i-1}, C_i, C_{i+1}, and C_i^{opp} – a cubic curve fitted onto the initial and (negated) end derivatives of sides $i+2$ and $i-2$, respectively (see Figure 1).

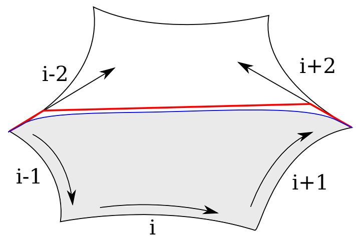


Figure 1: Construction of a four-sided Coons ribbon.

Formally,

$$\begin{aligned} R_i(s_i,d_i) &= (1-d_i)C_i(s_i) + d_i C_i^{\text{opp}}(1-s_i) \\ &\quad + (1-s_i)C_{i-1}(1-d_i) + s_i C_{i+1}(d_i) \\ &\quad - \begin{bmatrix} 1-s_i \\ s_i \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} C_i(0) & C_{i-1}(0) \\ C_i(1) & C_{i+1}(1) \end{bmatrix} \begin{bmatrix} 1-d_i \\ d_i \end{bmatrix}, \end{aligned} \tag{1}$$

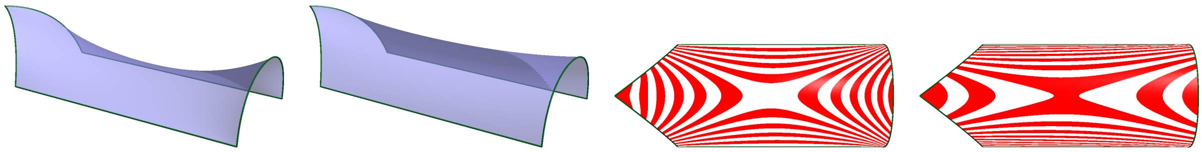
where C^{opp} is defined as the Bézier curve† determined by the control points

$$P_0 = C_{i+1}(1), \qquad P_1 = P_0 + \frac{1}{3}C'_{i+2}(0), \tag{2}$$

$$P_2 = P_3 - \frac{1}{3}C'_{i-2}(1), \qquad P_3 = C_{i-1}(0). \tag{3}$$

The surface is defined over a regular n-sided polygon. The

† Except for $n = 3$, where C^{opp} degenerates to the point $C_{i+1}(1)$.



(a) Harmonic surface (b) C^0 Coons patch (c) Harmonic surface (contours) (d) C^0 Coons patch (contours)

Figure 2: Comparison with the harmonic surface on a 5-sided boundary loop.

Wachspress coordinates of a domain point p are defined as

$$\lambda_i = \lambda_i(p) = \frac{\prod_{j \neq i-1,i} D_j(p)}{\sum_{k=1}^{n} \prod_{j \neq k-1,k} D_j(p)}, \quad (4)$$

where $D_i(p)$ is the perpendicular distance of p from the i-th edge of the domain polygon. Ribbon parameterization is based on these generalized barycentric coordinates:

$$d_i = d_i(u,v) = 1 - \lambda_{i-1} - \lambda_i, \quad s_i = s_i(u,v) = \frac{\lambda_i}{\lambda_{i-1} + \lambda_i}. \quad (5)$$

It is easy to see that $s_i, d_i \in [0,1]$, and that d_i has the following properties:

1. $d_i = 0$ on the i-th side.
2. $d_i = 1$ on the "far" sides (all sides except $i-1$, i and $i+1$).
3. $d_{i-1} + d_{i+1} = 1$ on the i-th side.

Finally, we define the patch as

$$S(p) = \sum_{i=1}^{n} R_i(s_i, d_i) B_i(d_i), \quad (6)$$

where B_i is the blending function

$$B_i(d_i) = \frac{1 - d_i}{2}. \quad (7)$$

The interpolation property is satisfied due to the properties of d_i mentioned above.

(Note: s_i in Eq. (5) cannot be evaluated when $d_i = 1$, but at these locations the weight $B_i(d_i)$ also vanishes.)

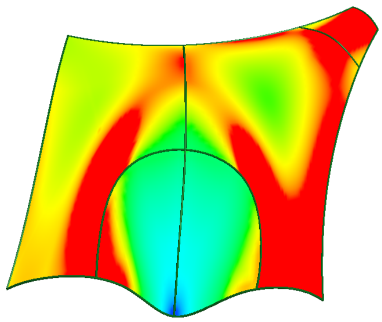
4. Examples

Figure 2 shows a comparison with the harmonic surface, which – due to its area minimizing property – results in an unnaturally flat patch.

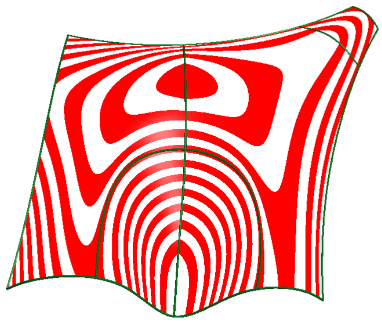
Figure 3 shows a model with 5 patches: two 3-sided, one 4-sided, one 5-sided, and one 6-sided. The mean curvature map and contouring both show good surface quality.

Conclusion

We have defined a natural generalization of the C^0 Coons patch – a lightweight and efficient multi-sided surface representation, applicable when only positional data is available.



(a) Mean curvature map



(b) Contouring

Figure 3: The "pocket" model.

Acknowledgements

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References

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